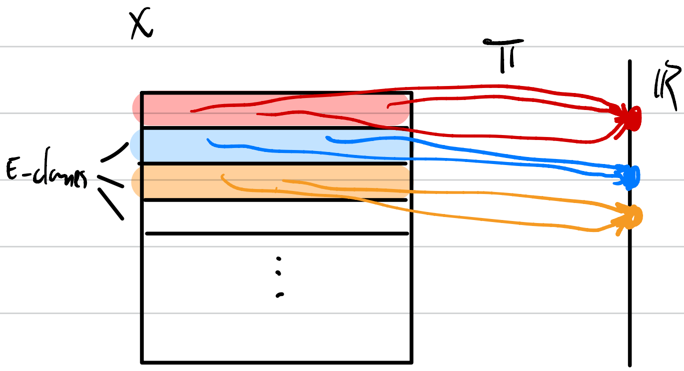
Ergodic Theory and Measured Group Theory
Lecture 18

Isomorphism and classification. Recall that two actions $\Gamma \curvearrowright^{\alpha} (X, \mu)$ and $\Gamma \curvearrowright^{\beta} (Y, \nu)$ of a semigroup Γ are called isomorphic if if $\exists$ measure-preserving Γ-equivariant map $\pi : X \to Y$ that is an almost-injection, i.e.

- measure-preserving: $\pi_* \mu = \nu$, in particular π is almost-surj.
- Γ-equivariant: $\gamma_\beta \circ \pi = \pi \circ \gamma_\alpha$,
- almost-injection: $\exists$ conull $X' \subseteq X$ s.t. $\pi|_{X'}$ is 1-1.

What is classification? Given class X of objects and an eq. relation E on X, we would like to



Def. Let X be standard Borel space and let E be an equiv. rel. on X. This E is called concretely classifiable or smooth if $\exists$ Borel map $\pi: X \to \mathbb{R}$ (any other Polish space) s.t. $\forall x, x' \in X$, $x \, E \, x' \iff \pi(x) = \pi(x')$. In other words, "$\Rightarrow$" says that π descends to $\overline{\pi}: X/_E \to \mathbb{R}$, "$\Leftarrow$" says that $\overline{\pi}$ is injective, so $\overline{\pi}: X/_E \hookrightarrow \mathbb{R}$.

To apply this definition, we need to first encode the actions $\Gamma \curvearrowright (X, \mu)$ into a st. Borel space (different for each Γ). Let's first do so for $\Gamma := \mathbb{Z}$ and pmp actions $\mathbb{Z} \curvearrowright (X, \mu)$. WLOG, restrict attention to the case where μ is nonatomic, so $(X, \mu) \cong_{\text{measure}} ([0,1], \lambda)$, so we can encode only actions on a fixed (X, μ). The $\mathbb{Z}$-actions on (X, μ) are identified with m.p. automorphisms of (X, μ).

Def. Let $\mathrm{Aut}(\mu)$ denote the group of m.p. automorphisms $T: X \to X$ of μ (T is an isom. $(X, \mu) \xrightarrow{\sim} (X, \mu)$), where we identify T and T' if they differ on a null set, i.e. $T =_\mu T'$.

Claim. For any $T \in \mathrm{Aut}(\mu)$, there is another $T' \in \mathrm{Aut}(\mu)$ $T =_\mu T'$ s.t. $T': X \to X$ is a bijection.

Proof. Let X_0 be a conull Borel set s.t. $T|_{X_0}$ is 1-1.
Then $[X \setminus X_0]_T = \bigcup_{n \in \mathbb{Z}} T^n (X \setminus X_0)$ is still null so
$X_1 := X \setminus [X \setminus X_0]_T$ is T-invariant & T is 1-1 on it,
hence $T(X_1) = X_1$. $\square$

For any $T, T' \in \mathrm{Aut}(\mu)$, $T \cong T'$ $\iff$ $\exists$ meas.-isom. $\pi : (X,\mu) \to (X,\mu)$ s.t. $T \circ \pi = \pi \circ T'$ $\iff$ $\exists \pi \in \mathrm{Aut}(\mu)$ $\pi \circ T' \circ \pi^{-1} = T$
$\iff$ T & T' are conjugate.

So we translated the isomorphism relation of pmp actions of $\mathbb{Z}$ to the conjugacy relation in $\mathrm{Aut}(\mu)$.
We equip $\mathrm{Aut}(\mu)$ with a Polish topology making it a Polish group.

Strong topology. $d'(T_1, T_2) := \mu \{ x \in X : T_1 x \neq T_2 x \}$ is a metric, which defines a completely metrizable top. on $\mathrm{Aut}(\mu)$. Indeed, the metric $d(T_1, T_2) := d'(T_1, T_2) + d'(T_1^{-1}, T_2^{-1})$ is complete & induces the same topology. But the metric d' has the advantage of being left & right invariant:
$d'(T_1 \circ S, T_2 \circ S) = d'(T_1, T_2) = d'(S \circ T_1, S \circ T_2)$.

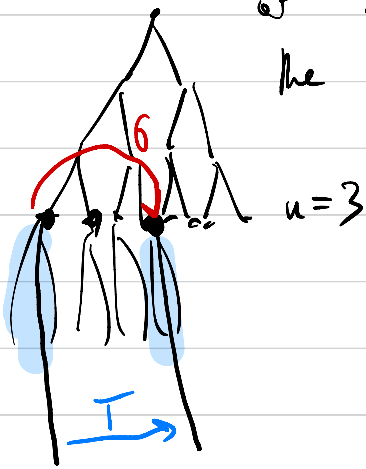
The downside of this topology is that it's too fine: it is not separable.

Weak topology. For each Borel $A \subseteq X$, $d_A(T_1, T_2) := \mu(T_1 A \triangle T_2 A)$. This is a pseudo-metric. The weak topology on $\mathrm{Aut}(\mu)$ is the one generated by all these d_A, $A \in \mathcal{B}(X)$, i.e. generated by sets $B_A(T_0, r) := \{T \in \mathrm{Aut}(\mu) : d_A(T, T_0) < r\}$. Thus, $T_n \to_{\text{weakly}} T \iff \forall A \in \mathcal{B}(X)$, $d_A(T_n, T) \to 0$.

Prop. The strong top. is metrizable with $\bar{d}(T_1, T_2) := \sup_{A \in \mathcal{B}(X)} d_A(T_1, T_2)$.

But the weak topology too is completely metrizable:
$$d_w'(T_1, T_2) := \sum_{n=1}^{\infty} 2^{-n} d_{U_n}(T_1, T_2),$$
where $(U_n)_{n \in \mathbb{N}}$ is a ctbl generating algebra of Borel sets. Then this is a compatible metric with the weak top and it has complete version: $d_w(T_1, T_2) := d_w'(T_1, T_2) + d_w'(T_1^{-1}, T_2^{-1})$.
The weak topology is separable: the set of dyadic permutations is dense.

Dyadic permutation. Let $X = 2^{\mathbb{N}}$. $T : X \to X$ is a dyadic permutation if $\exists n \in \mathbb{N}$ and a permutation σ of 2^n s.t. $\forall w \in 2^n$ and $x \in 2^{\mathbb{N}}$,
$T(wx) = \sigma(w)x$. In other words, thinking of $X = [0,1)$, T is a finite permutation of the intervals with dyadic endpoints.



Long story short: $\mathrm{Aut}(\mu)$ with the weak topology is a Polish group. Recall that isom of pmp actions of $\mathbb{Z}$ is the same as the conjugacy rel. $\sim$ on $\mathrm{Aut}(\mu)$. Now we can ask: is $\sim$ concretely classifiable?

Obs. If an eq. rel. E on a st. Borel space X is smooth, then E, as a subset of X^2, is Borel.

Proof. Let $\pi : X \to \mathbb{R}$ be witnessing smoothness. Then $\pi_2 : X^2 \to \mathbb{R}^2$ by $(x,y) \mapsto (\pi(x), \pi(y))$ is Borel being a composition of Borel maps, and $\pi_2^{-1}(=_{\mathbb{R}}) = E$, where $=_Y := \{(y,y) : y \in Y\}$, so E is Borel. $\square$

Question (von Neumann). Is $\sim$ on $\mathrm{Aut}(\mu)$ concretely classifiable?

Answer (Rudolph–Foreman–Weiss). No, in fact $\sim$ relation is not Borel (even restricted to a subgroup of $\mathrm{Aut}(\mu)$ of ergodic automorphisms, which is a G_δ subset).

Remarks. (a) A generic $T \in \mathrm{Aut}(\mu)$ is ergodic, more precisely, the set of ergodic T is dense G_δ $\Rightarrow$ comeager.

(b) It is still possible that if we restrict to a smaller (meager) subset of $\mathrm{Aut}(\mu)$, $\sim$ is smooth on that set.